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Topological Strings, Instantons and Asymptotic Forms of Gopakumar–Vafa Invariants

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Abstract

We calculate the topological string amplitudes of Calabi–Yau toric threefolds corresponding to $4D$, $\mathcal{N} = 2$ $SU(2)$ gauge theory with $N_f = 0, 1, 2, 3, 4$ fundamental hypermultiplets by using the method of the geometric transition and show that they reproduce Nekrasov’s formulas for instanton counting. We also determine the asymptotic forms of the Gopakumar–Vafa invariants of the Calabi–Yau threefolds including those at higher genera from instanton amplitudes of the gauge theory.

1 Introduction

Recently, remarkable developments occurred in the theory of the topological strings. We can now compute the Gromov–Witten invariants or Gopakumar–Vafa invariants of a Calabi–Yau toric threefold at all genera by using the Feynman-like rules [1, 2] which has been developed from the geometric transition and the Chern–Simons theory [3]. Although this method is most powerful compared to other methods such as localization and local B-model calculation, we still cannot obtain the exact form of the topological string amplitude in general cases because we have to sum over several partitions. However, it is found that we can perform the summation for some special types of tree graphs by using the identities on the skew-Schur functions [4, 5, 6]. A simplest example is the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbf{P}^1$.

Another interesting application is the geometric engineering of the gauge theories [7, 8, 4, 5]. In this article we study the cases with the gauge group $SU(2)$ and with $N_f = 0, 1, 2, 3, 4$ fundamental hypermultiplets. It has been known that the corresponding Calabi–Yau threefolds are the canonical bundles of the Hirzebruch surfaces blown up at N_f -points. We calculate the topological string amplitudes of these Calabi–Yau threefolds and show that they reproduce Nekrasov’s formulas for instanton counting [9] in a certain limit. The $SU(n+1)$ cases without hypermultiplets and the $SU(2)$ case with one hypermultiplet have been studied in [7, 8, 4] and the calculations in this article are essentially the same. We also determine the asymptotic forms of the Gopakumar–Vafa invariants of these Calabi–Yau threefolds from the relation between the topological string amplitudes and Nekrasov’s formula. This result is the generalization of the genus zero results [10, 11] to higher genus cases.

The organization of the paper is as follows. In section 2, we calculate the topological string amplitudes of Calabi–Yau toric threefolds that correspond to $4D$, $\mathcal{N} = 2$ $SU(2)$ gauge theories with $N_f = 0, 1, 2, 3, 4$ fundamental hypermultiplets. In section 3, we show that the topological amplitude reproduces Nekrasov’s formula. In section 4, we derive the asymptotic form of the Gopakumar–Vafa invariants of the Calabi–Yau toric threefolds. Appendices contain formulas and the calculation of the framing.

2 Topological String Amplitude

In this section, we calculate the topological string amplitudes of Calabi–Yau toric threefolds that reproduce four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories with gauge group $SU(2)$ and with $N_f = 0, 1, 2, 3, 4$ fundamental hypermultiplets.

First we briefly review the calculation of the topological string amplitudes of Calabi–Yau threefolds X following [2] when X is the canonical bundle of a smooth toric surface classified in [12]. Recall that a two-dimensional integral polytope (the section of the fan at the height 1) of X has only one interior integral point $(0, 0)$ and this point and each integral point $v_i (1 \leq i \leq k)$ on the boundary span an interior edge (here we define k to be the number of the interior edges and take v_1, v_2, \dots in the clockwise direction). Therefore the corresponding web diagram consists of a polygon with k -edges and external lines attached to it. We take the orientation of edges on the polygon in the clockwise direction, and that of the external lines in the outgoing direction. Then the integer m_i of the framing for an interior edge dual to v_i is given by

$$m_i = -\gamma_i - 1. \quad (1)$$

Here γ_i is the self-intersection number of the \mathbf{P}^1 associated to v_i and computed from the equation

$$-\gamma_i v_i = v_{i-1} + v_{i+1}. \quad (2)$$

The derivation of m_i is included in appendix. Then we assign a partition Q_i to each interior edge and the partition of zero to each external edge. Finally we obtain the topological string amplitude by multiplying all quantities associated to vertices and edges and by summing over all partitions. The brief summary of the rule is as follows: to a trivalent vertex we associate the three-point amplitude C_{R_1, R_2, R_3} if the orientation of all the edges are outgoing, $(-1)^{l(R_1)} C_{R_1^t, R_2, R_3}$ if the orientation of one edge with a partition R_1 is incoming, etc, where R_1, R_2, R_3 are partitions assigned to three edges attached to the vertex; to an interior edge we associate $(-1)^{ml(R)} q^{-\frac{m\kappa(R)}{2}} e^{-l(R)t}$ where m is the integer coming from the framing and R is the partition assigned to the edge and t is the Kähler parameter of the corresponding \mathbf{P}^1 . Here $l(R) := \sum_i \mu_i$ for a partition $R = (\mu_1, \mu_2, \dots)$ and $\kappa(R) := \sum_i \mu_i(\mu_i - 2i + 1)$. Thus

the topological string amplitude for X is written as

$$\mathcal{Z} = \sum_{Q_1, \dots, Q_k} \prod_{i=1}^k C_{Q_i^t, \emptyset, Q_{i+1}} (-1)^{\gamma_i l(Q_i)} q^{\frac{\gamma_i+1}{2} \kappa(Q_i)}. \quad (3)$$

Here we have defined $\gamma_{k+1} := \gamma_1, Q_{k+1} := Q_1$. This result was derived in [13, 14, 4]. The relation between the topological string amplitude \mathcal{Z} and the Gromov–Witten invariants is that $\log \mathcal{Z}|_{q=e^{\sqrt{-1}g_s}}$ is the generating function of the Gromov–Witten invariants where g_s is the genus expansion parameter. This statement has been proved for the canonical bundle of fano toric surfaces [14]¹.

Next we derive more compact formulas for the toric Calabi–Yau threefolds that corresponding to $4D$, $\mathcal{N} = 2$ $SU(2)$ gauge theories with $N_f = 0, 1, 2, 3, 4$ fundamental hypermultiplets. The Calabi–Yau threefolds are the canonical bundles of the Hirzebruch surfaces $\mathbf{F}_0, \mathbf{F}_1$, or \mathbf{F}_2 blown up at N_f points. There exist 3,2,3,3,2 such Calabi–Yau threefolds for $N_f = 0, 1, 2, 3, 4$.² The fans and the web diagrams for these threefolds are shown in figures 1 and 3. We take $[C_B], [C_F], [C_{E_i}]$ ($1 \leq i \leq N_f$) as a basis of the second homology where C_B (resp. C_F) is the base \mathbf{P}^1 (resp. fiber \mathbf{P}^1) and C_{E_i} is an exceptional curve. The intersections are

$$\begin{aligned} C_B \cdot C_B &= -b, & C_B \cdot C_F &= 1, & C_F \cdot C_{E_i} &= 0, \\ C_F \cdot C_F &= 0, & C_F \cdot C_{E_i} &= 0, & C_{E_i} \cdot C_{E_j} &= -\delta_{i,j} \end{aligned} \quad (4)$$

for $1 \leq i, j \leq N_f$. The values of b are 1 or 2 and will be listed later. Here t_B, t_F, t_{E_i} ($1 \leq i \leq 3$) denote the Kähler parameters of the base \mathbf{P}^1 , the fiber \mathbf{P}^1 , the i -th (-1) -curve and $q_B := e^{-t_B}, q_F := e^{-t_F}, q_i := e^{-t_{E_i}}$.

Now we compute the topological string amplitudes by using the same strategy as [7, 8, 4]. Let us take $N_f = 2$ cases shown in figure 1 as examples. We first cut the polygon in the web diagram into two upper and lower parts (as shown by dotted line) and compute the amplitudes separately. Then we glue the two amplitudes together along the vertical edges

¹ \mathbf{P}^2 and (2)(3)(5)(7) in figures 1, 3.

²Although there are 16 smooth toric surfaces classified in [12], three among them (1,10,16 in figure 1 in [12]) do not correspond to the four-dimensional gauge theories.

$(1 \leq k \leq 4)$ are amplitudes corresponding to the web diagrams in figure 2 and defined by

$$H_{R_1, R_2}^{(1)}(x) := \sum_Q q^{\frac{\kappa(Q)}{2}} x^{l(Q)} (-1)^{l(R_2)} C_{\emptyset, R_1, Q^t} C_{Q, R_2^t, \emptyset}, \quad (8)$$

$$H_{R_1, R_2}^{(k)}(x_1, \dots, x_k) := \sum_{Q_1, \dots, Q_k} (-1)^{l(R_2) + l(Q_1) + l(Q_k)} q^{-\frac{\kappa(Q_2) + \dots + \kappa(Q_{k-1})}{2}} \prod_{i=1}^k x_i^{l(Q_i)} \quad (9)$$

$$\times C_{\emptyset, R_1, Q_1^t} C_{Q_2^t, \emptyset, Q_1} \cdots C_{Q_k^t, \emptyset, Q_{k-1}} C_{Q_k, R_2^t, \emptyset} \quad (2 \leq k \leq 4).$$

Using the expression for C_{R_1, R_2, R_3} written in terms of the skew-Schur functions and the identities (see appendix), $H_{R_1, R_2}^{(k)}(x_1, \dots, x_k)$ becomes

$$H_{R_1, R_2}^{(k)}(x_1, \dots, x_k) = (-1)^{l(R_2)} W_{R_1} W_{R_2^t} K(x_1 \cdots x_k) g_{R_1, R_2}(x_1 \cdots x_k) \times \left[\prod_{j=1}^{k-1} K(x_1 \cdots x_j) g_{R_1, \emptyset}(x_1 \cdots x_j) \prod_{j=2}^k K(x_j \cdots x_k) g_{R_2, \emptyset}(x_j \cdots x_k) \right]^{-1} \times \prod_{2 \leq i \leq j \leq k-1} K(x_i \cdots x_j). \quad (10)$$

The second (resp. third) line should be set to 1 for $k = 1$ (resp. for $k \leq 2$).

$$K(x) := \sum_{i,j=1}^{\infty} (1 - x^{-i-j+1})^{-1} = \exp \left[\sum_{k=1}^{\infty} \frac{q^k x^k}{k(q^k - 1)} \right] \quad (11)$$

$$W_R := q^{\frac{\kappa R}{4}} \prod_{1 \leq i < j \leq d(R)} \frac{[\mu_i - \mu_j + j - i]}{[j - i]} \prod_{i=1}^{d(R)} \prod_{v=1}^{\mu_i} \frac{1}{[v - u + d(R)]}, \quad (12)$$

$$g_{R_1, R_2}(x) := \prod_{i,j=1}^{\infty} \left[\frac{(1 - x q^{\mu_{1,i} - i + \mu_{2,j} - j + 1})}{(1 - x q^{-i-j+1})} \right] \quad (13)$$

$$= \prod_{(i,j) \in R_1} \frac{1}{(1 - x q^{\mu_{1,i} - j + \mu_{2,j} - i + 1})} \prod_{(j,i) \in R_2} \frac{1}{(1 - x q^{-\mu_{1,j}^{\vee} + i - \mu_{2,i}^{\vee} + j - 1})}. \quad (14)$$

Here $[k] := q^{\frac{k}{2}} - q^{-\frac{k}{2}}$, $l(R) := \sum_i \mu_i$ for a partition $R = (\mu_1, \mu_2, \dots)$, $\kappa(R) := \sum_i \mu_i(\mu_i - 2i + 1)$ and $d(R)$ is the length of R . $(\mu_{1,i}^{\vee})_{i \geq 1}$ (resp. $(\mu_{2,i}^{\vee})_{i \geq 1}$) is the conjugate partition of R_1 (resp. R_2) and $(i, j) \in R$ means that there is a box in the place of i -th row and j -th column in the partition R regarded as a Young diagram. We have used an identity in appendix to obtain the expression (14). The final form of the topological string amplitudes for the $N_f = 2$ cases

become

$$\begin{aligned}
\mathcal{Z} &= Z_0 Z_{\geq 1} \\
Z_0 &= K(q_F)^2 \prod_{j=1}^{N_f} K(q_j)^{-1} K(q_F q_j^{-1})^{-1} \times \begin{cases} 1 & (7) \\ K(q_1 q_2^{-1}) & (8)(9) \end{cases} \\
Z_{\geq 1} &= \sum_{R_1, R_2} g_{R_1, R_1^t}(1) g_{R_2, R_2^t}(1) g_{R_1, R_2}(q_F)^2 \prod_{j=1}^{N_f} g_{R_1, \emptyset}(q_j)^{-1} g_{R_2, \emptyset}(q_F q_j^{-1})^{-1} \\
&\quad \times q_B^{l(R_1)+l(R_2)} q_F^{bl(R_1)} (q_1 \cdots q_{N_f})^{-l(R_1)} (-1)^{m_1 l(R_1) + m_2 l(R_2)} q^{-\frac{m_1 \kappa(R_1) + m_2 \kappa(R_2)}{2}}.
\end{aligned} \tag{15}$$

The numbers b are 1, 2, 1 and (m_1, m_2) are $(0, 0)$, $(-1, 1)$, $(0, 0)$ for (7)(8)(9). In $Z_{\geq 1}$ we have used the identity

$$W_R W_{R^t} = (-1)^{l(R)} g_{R_1, R_1^t}(1).$$

The generating function of the Gromov–Witten invariants is obtained from the topological string amplitude by taking the logarithm and substituting $e^{\sqrt{-1}g_s}$ into q :

$$\begin{aligned}
\log \mathcal{Z}|_{q=e^{\sqrt{-1}g_s}} &= \sum_{g=0}^{\infty} g_s^{2g-2} \sum_{d_B, d_F, d_1, d_2} N_{g, d_B, d_F, d_1, d_2} q_B^{d_B} q_F^{d_F} q_1^{d_1} q_2^{d_2} \\
&= \sum_{g=0}^{\infty} \sum_{d_B, d_F, d_1, d_2} \sum_{k=1}^{\infty} \frac{n_{d_B, d_F, d_1, d_2}^g}{k} \left(2 \sin \frac{k g_s}{2}\right)^{2g-2} (q_B^{d_B} q_1^{d_1} q_F^{d_F} q_1^{d_1} q_2^{d_2})^k.
\end{aligned} \tag{16}$$

$N_{g, d_B, d_F, d_1, d_2}$ denotes the genus zero, 0-pointed Gromov–Witten invariant for an integral homology class $d_B[C_B] + d_F[C_F] + d_1[C_{E_1}] + d_2[C_{E_2}]$ and n_{d_B, d_F, d_1, d_2}^g denotes the Gopakumar–Vafa invariant.

One can read off the Gromov–Witten invariants with $d_B = 0$ from Z_0 , because only $\log Z_0$ gives the terms with degree zero in q_B :

$$\begin{aligned}
\log Z_0 &= 2 \sum_{k=1}^{\infty} \frac{q^k q_F^k}{k(q^k - 1)^2} - \sum_{i=1}^{N_f} \left[\sum_{k=1}^{\infty} \frac{q^k q_i^k}{k(q^k - 1)^2} + \sum_{k=1}^{\infty} \frac{q^k (q_F q_i^{-1})^k}{k(q^k - 1)^2} \right] \\
&\quad + \sum_{k=1}^{\infty} \frac{q^k (q_1 q_2^{-1})^k}{k(q^k - 1)^2} \text{ for (8)(9), 0 for (7)} \\
&\stackrel{q=e^{\sqrt{-1}g_s}}{=} \sum_{k=1}^{\infty} \frac{1}{k} \left(2 \sin \frac{k g_s}{2}\right)^{-2} \left[-2q_F^k + \sum_{i=1}^{N_f} (q_F^k + (q_F q_i^{-1})^k) \right] \\
&\quad + \sum_{k=1}^{\infty} \frac{-1}{k} \left(2 \sin \frac{k g_s}{2}\right)^{-2} (q_1 q_2^{-1})^k \text{ for (8)(9), 0 for (7).}
\end{aligned} \tag{17}$$

Hence the nonzero Gopakumar–Vafa invariants for a second homology class $d_B[C_B] + d_F[C_F] + d_1[C_{E_1}] + d_2[C_{E_2}]$ with $d_B = 0$ are as follows (we slightly change the notation and write the Gopakumar–Vafa invariant as n_α^g for a second homology class α):

$$n_{[C_F]}^{g=0} = -2, \quad n_{[C_{E_i}]}^{g=0} = n_{[C_F] - [C_{E_i}]}^{g=0} = +1 \quad (1 \leq i \leq N_f), \quad (18)$$

and

$$n_{[C_{E_1}] - [C_{E_2}]}^{g=0} = -1 \text{ only for } (8)(9). \quad (19)$$

The invariants at $[C_F]$, $[C_{E_i}]$ and $[C_F] - [C_{E_i}]$ ($1 \leq i \leq N_f$) are the same in all of the three cases, while the invariant at $[C_{E_1}] - [C_{E_2}]$ are different. We will interpret these results from the viewpoint of relation to the Seiberg–Witten prepotential of $SU(2)$ gauge theory in the next section.

The Gopakumar–Vafa invariants with $d_B \geq 1$ are given by $\log Z_{\geq 1}$. We remark two properties. The one is that the Gopakumar–Vafa invariant is nonzero only when

$$0 \leq -d_i \leq d_B \quad (1 \leq \forall i \leq N_f). \quad (20)$$

One could read this fact from the expression (15) as follows. The summand is the polynomial in q_1^{-1}, q_2^{-1} of degree at most $l(R_1) + l(R_2)$ given that $g_{R,\emptyset}(x)^{-1}$ is the polynomial in x of degree $l(R)$. Therefore the degree in q_i^{-1} is always equal or smaller than the degree in q_B , and this fact persists when we take the logarithm. Thus the Gromov–Witten invariants are zero unless the condition (20) is satisfied, and so are the Gopakumar–Vafa invariants. The other property is that the Gopakumar–Vafa invariants are symmetric with respect to q_1, q_2 . This follows from the invariance of $Z_{\geq 1}$ under the exchange of q_1 and q_2 .

Finally, we summarize the topological string amplitude for all the cases corresponding to $SU(2)$ gauge theory with $N_f = 0, 1, 2, 3, 4$ hypermultiplets shown in figures 1, 3.

$$\begin{aligned} \mathcal{Z} &= Z_0 Z_{\geq 1} \\ Z_0 &= K(q_F)^2 \prod_{j=1}^{N_f} K(q_j)^{-1} K(q_F q_j^{-1})^{-1} \times (a) \\ Z_{\geq 1} &= \sum_{R_1, R_2} g_{R_1, R_1^t}(1) g_{R_2, R_2^t}(1) g_{R_1, R_2}(q_F)^2 \prod_{j=1}^{N_f} g_{R_1, \emptyset}(q_j)^{-1} g_{R_2, \emptyset}(q_F q_j^{-1})^{-1} \\ &\quad \times q_B^{l(R_1) + l(R_2)} q_F^{bl(R_1)} (q_1 \cdots q_{N_f})^{-l(R_1)} (-1)^{m_1 l(R_1) + m_2 l(R_2)} q^{-\frac{m_1 \kappa(R_1) + m_2 \kappa(R_2)}{2}}. \end{aligned} \quad (21)$$

N_f		Nonzero Gopakumar–Vafa invariants
0	(2)(3)(4)	$n_{[C_F]}^{g=0}$
1	(5)(6)	$n_{[C_F]}^{g=0} = -2, n_{[C_{E_1}]}^{g=0} = n_{[C_F]-[C_{E_1}]}^{g=0} = 1$
2	(7)	$n_{[C_F]}^{g=0} = -2, n_{[C_{E_i}]}^{g=0} = n_{[C_F]-[C_{E_i}]}^{g=0} = 1 (1 \leq i \leq 2)$
	(8)(9)	$n_{[C_F]}^{g=0} = -2, n_{[C_{E_i}]}^{g=0} = n_{[C_F]-[C_{E_i}]}^{g=0} = 1 (1 \leq i \leq 2), n_{[C_{E_1}]-[C_{E_2}]} = -1$
3	(11)(13)	$n_{[C_F]}^{g=0} = -2, n_{[C_{E_i}]}^{g=0} = n_{[C_F]-[C_{E_i}]}^{g=0} = 1 (1 \leq i \leq 3),$ $n_{[C_{E_1}]-[C_{E_2}]} = n_{[C_{E_1}]-[C_{E_3}]} = n_{[C_{E_2}]-[C_{E_3}]} = -1$
	(12)	$n_{[C_F]}^{g=0} = -2, n_{[C_{E_i}]}^{g=0} = n_{[C_F]-[C_{E_i}]}^{g=0} = 1 (1 \leq i \leq 3), n_{[C_{E_1}]-[C_{E_2}]} = -1$
4	(14)	$n_{[C_F]}^{g=0} = -2, n_{[C_{E_i}]}^{g=0} = n_{[C_F]-[C_{E_i}]}^{g=0} = 1 (1 \leq i \leq 4),$ $n_{[C_{E_1}]-[C_{E_2}]} = n_{[C_{E_1}]-[C_{E_3}]} = n_{[C_{E_2}]-[C_{E_3}]} = -1$
	(15)	$n_{[C_F]}^{g=0} = -2, n_{[C_{E_i}]}^{g=0} = n_{[C_F]-[C_{E_i}]}^{g=0} = 1 (1 \leq i \leq 4), n_{[C_{E_1}]-[C_{E_2}]} = n_{[C_{E_3}]-[C_{E_4}]} = -1$

Table 1: The nonzero Gopakumar–Vafa invariants for a second homology class $d_B[C_B] + d_F[C_F] + d_1[C_{E_1}] + d_{N_f}[C_{E_{N_f}}]$ with $d_B = 0$.

where

$$(a) = \begin{cases} 1 & (2)(3)(4)(5)(6)(7) \\ K(q_1 q_2^{-1}) & (8)(9)(12) \\ K(q_1 q_2^{-1}) K(q_1 q_3^{-1}) K(q_2 q_3^{-1}) & (11)(13)(14) \\ K(q_1 q_2^{-1}) K(q_3 q_4^{-1}) & (15). \end{cases}$$

b is the self-intersection of the base \mathbf{P}^1 and $m_1(m_2)$ is the integer of the framing of the right (left) vertical edge:

	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(11)	(12)	(13)	(14)	(15)
N_f	0			1		2			3			4	
b	0	1	2	1	2	1	2	1	2	2	1	1	1
m_1	-1	-2	-3	-1	-2	0	-1	0	1	0	0	1	1
m_2	-1	0	1	0	1	0	1	0	0	1	1	1	1

Note that the properties of the Gopakumar–Vafa invariants mentioned in $N_f = 2$ cases hold in all N_f cases: the nonzero Gopakumar–Vafa invariants for a second homology class $d_B[C_B] + d_F[C_F] + d_1[C_{E_1}] + d_{N_f}[C_{E_{N_f}}]$ with $d_B = 0$ are summarized in table 1 and one can

easily see that in all cases $n_{[C_F]}^0 = -2$, $n_{[C_{E_i}]}^0 = n_{[C_F] - [C_{E_i}]}^0 = 1$ ($1 \leq i \leq N_f$); for $d_B \geq 1$, the Gopakumar–Vafa invariants is nonzero only if $0 \leq -d_i \leq d_B$ ($1 \leq i \leq N_f$) and they are symmetric with respect to d_1, \dots, d_{N_f} . Note also that $Z_{\geq 1}$'s of (7) and (9) (resp. (14) and (15)) are the same, which means that the Gopakumar–Vafa invariants in cases (7) and (9) (resp. (14) and (15)) with $d_B \geq 1$ are completely the same.

3 Nekrasov's Formula

In this section we show that the topological string amplitude \mathcal{Z} gives the one-loop corrections in the prepotential and Nekrasov's formula [9] for the $4D$, $\mathcal{N} = 2$ $SU(2)$ gauge theory with N_f fundamental hypermultiplets at a certain limit. The argument in this section closely follows that in [7, 8, 4].

First let us identify the parameters in the two sides:

$$t_F = -2\beta a, \quad t_{E_i} = -\beta(a + m_i) \quad (1 \leq i \leq N_f), \quad q = -\beta\hbar. \quad (22)$$

Then the limit we should take is the limit $\beta \rightarrow 0$. Here $a = a_1 = -a_2$ is the vacuum expectation value of the complex scalar field in a gauge multiplets, m_i ($1 \leq i \leq N_f$)'s are mass parameters of the fundamental hypermultiplets. g_s has been introduced as the genus expansion parameter.

Next, let us show that the t_B independent part Z_0 in the topological string amplitude gives the perturbative one-loop correction terms. We again take $N_f = 2$ cases (7)(8)(9) as examples. Note that

$$\lim_{\hbar \rightarrow 0} \frac{q^k}{(q^k - 1)^2} \Big|_{q=e^{-\beta\hbar}} = \frac{1}{\beta^2 \hbar^2}.$$

Then

$$\begin{aligned} \lim_{\hbar \rightarrow 0} \hbar^2 \log Z_{\geq 1} &= 2 \sum_{k=1}^{\infty} \frac{1}{k^3} \left[e^{-2k\beta a} - \sum_{i=1}^{N_f} \left(e^{-\beta(a+m_i)k} + e^{-\beta(a-m_i)k} \right) \right] \\ &+ \sum_{k=1}^{\infty} \frac{1}{k^3} e^{-k\beta(m_1-m_2)} \text{ for (8)(9), } 0 \text{ for (7).} \end{aligned} \quad (23)$$

Each trilogarithm corresponds to one logarithmic term in the Seiberg–Witten prepotential. We can see this correspondence in the following way. If we take the third derivative in a ,

the right-hand side of (23) becomes

$$\begin{aligned}
& -\frac{8e^{-2\beta a}}{1-e^{-2\beta a}} + \sum_{i=1}^{N_f} \left(\frac{e^{-\beta(a+m_i)}}{1-e^{-\beta(a+m_i)}} + \frac{e^{-\beta(a-m_i)}}{1-e^{-\beta(a-m_i)}} \right) \\
& \stackrel{\beta \rightarrow 0}{=} -\frac{8}{a} + \sum_{i=1}^{N_f} \left(\frac{1}{a+m_i} + \frac{1}{a-m_i} \right).
\end{aligned} \tag{24}$$

In passing to the second line, we have used the formula $\sum_{k=0}^{\infty} x^k = (1-x)^{-1}$. This is exactly the third derivative of the one-loop correction in the $SU(2)$ Seiberg–Witten prepotential with N_f fundamental hypermultiplets. Note that the last term in (23) does not cause any problem because such term depends only on mass parameters, not on a .

Now we show that the logarithm of t_B -dependent part $Z_{\geq 1}$ in (21) gives the instanton correction terms in the gauge theory. More precisely we show that $Z_{\geq 1}$ becomes Nekrasov's formula in the limit $\beta \rightarrow 0$. We introduce the following function for the sake of convenience: for two partitions $R_1 = (\mu_{1,i})_{i \geq 1}$ and $R_2 = (\mu_{2,i})_{i \geq 1}$,

$$\begin{aligned}
P_{R_1, R_2}(x) &:= \prod_{(i,j) \in R_1} \frac{1}{\sinh \frac{\beta}{2} (\tilde{a} + \hbar(\mu_{1,i} - j + \mu_{2,j}^\vee - i + 1))} \\
&\times \prod_{(i,j) \in R_2} \frac{1}{\sinh \frac{\beta}{2} (\tilde{a} + \hbar(-\mu_{1,j}^\vee + i - \mu_{2,i} + j - 1))},
\end{aligned} \tag{25}$$

where \tilde{a}, \hbar, β are defined by $q = e^{-\beta \hbar}$ and $x = e^{-\beta \tilde{a}}$. Then the following equations hold:

$$\begin{aligned}
P_{R_1, R_2}(x^{-1}) &= (-1)^{l(R_1) + l(R_2)} P_{R_2, R_1}(x), \\
P_{R_1^t, R_2^t}(x, q) &= P_{R_2, R_1}(x, q), \\
g_{R_1, R_2}(x) &= x^{-\frac{l(R_1) + l(R_2)}{2}} 2^{-l(R_1) - l(R_2)} q^{\frac{-\kappa(R_1) - \kappa(R_2)}{4}} P_{R_1, R_2^t}(x), \\
P_{R_1, R_2}(x) &= \prod_{k, l=1}^{\infty} \frac{\sinh \frac{\beta}{2} (\tilde{a} + \hbar(\mu_{1,k} - \mu_{2,l} + l - k))}{\sinh \frac{\beta}{2} (\tilde{a} + \hbar(l - k))}.
\end{aligned} \tag{26}$$

Therefore we can rewrite $Z_{\geq 1}$ as follows.

$$\begin{aligned}
Z_{\geq 1} &= \sum_{R_1, R_2} P_{R_1, R_2}(q_F) P_{R_2, R_1}(q_F^{-1}) P_{R_1, R_1}(1) P_{R_2, R_2}(1) \\
&\times \prod_{j=1}^{N_f} P_{R_1, \emptyset}(q_j) P_{R_2, \emptyset}(q_j^{-1}) \\
&\times 2^{-(4-N_f)(l(R_1)+l(R_2))} q_B^{l(R_1)+l(R_2)} q_F^{bl(R_1)-2l(R_1)-2l(R_2)} \prod_{j=1}^{N_f} q_j^{\frac{-l(R_1)+l(R_2)}{2}} \\
&\times (-1)^{m_1 l(R_1)+m_2 l(R_2)+N_f l(R_2)} q^{-\frac{m_1 \kappa(R_1)-m_2 \kappa(R_2)}{2} + \frac{(2-N_f)(-\kappa(R_1)+\kappa(R_2))}{4}}. \\
&= \sum_{R_1, R_2} \prod_{i,j=1,2} \prod_{k,l=1}^{\infty} \frac{\sinh \frac{\beta}{2}(a_{i,j} + \hbar(\mu_{i,k} - \mu_{j,l} + l - k))}{\sinh \frac{\beta}{2}(a_{i,j} + \hbar(l - k))} \\
&\times \prod_{j=1}^{N_f} \prod_{(k,l) \in R_1} \sinh \frac{\beta}{2}(a_1 + m_j + \hbar(k - l)) \prod_{(k,l) \in R_2} \sinh \frac{\beta}{2}(a_2 + m_j + \hbar(k - l)) \\
&\times 2^{-(4-N_f)(l(R_1)+l(R_2))} q_B^{l(R_1)+l(R_2)} q_F^{bl(R_1)-2l(R_1)-2l(R_2)} \prod_{j=1}^{N_f} q_j^{\frac{-l(R_1)+l(R_2)}{2}} \\
&\times (-1)^{m_1 l(R_1)+m_2 l(R_2)+N_f l(R_2)} q^{-\frac{m_1 \kappa(R_1)-m_2 \kappa(R_2)}{2}}.
\end{aligned} \tag{28}$$

We have rewritten R_2 in the summation as R_2^t and $a_{1,2} = a_1 - a_2 = 2a = -a_{2,1}$. Next we look into the limit $\beta \rightarrow 0$. The first line of (28) becomes

$$\left(\frac{\beta}{2}\right)^{-4(l(R_1)+l(R_2))} \prod_{i,j=1,2} \prod_{k,l=1}^{\infty} \frac{a_{i,j} + \hbar(\mu_{i,k} - \mu_{j,l} + l - k)}{a_{i,j} + \hbar(l - k)}.$$

The second line becomes

$$\hbar^{N_f(l(R_1)+l(R_2))} \left(\frac{\beta}{2}\right)^{N_f(l(R_1)+l(R_2))} \prod_{j=1}^{N_f} \prod_{(k,l) \in R_1} \left(\frac{a_1 + m_j}{\hbar} + (l - k)\right) \prod_{(k,l) \in R_2} \left(\frac{a_2 + m_j}{\hbar} + (l - k)\right).$$

The third line and the fourth become

$$2^{-(4-N_f)(l(R_1)+l(R_2))} q_B^{l(R_1)+l(R_2)}, \quad (-1)^{c(l(R_1)+l(R_2))}$$

where

$$c = \begin{cases} 1 & \text{for } (2)(4)(5)(8)(9)(11)(14)(15) \\ 0 & \text{for } (3)(6)(7)(12)(13). \end{cases} \tag{29}$$

Thus if we take the limit $\beta \rightarrow 0$ with

$$\begin{aligned} q_B &= (-1)^c \beta^{4-N_f} \tilde{q}, \\ t_F &= -2\beta a, \quad t_{E_i} = -\beta(a + m_i) \quad (1 \leq i \leq N_f), \quad q = -\beta \hbar, \end{aligned} \quad (30)$$

the topological string amplitude (21) becomes Nekrasov's formula for instanton counting [9]:

$$\begin{aligned} \lim_{\beta \rightarrow 0} \log Z_{\geq 1} &= \mathcal{Z}_{\text{Nekrasov}}^{(N_f)} \\ &= \sum_{R_1, R_2} (\tilde{q} \hbar^{N_f})^{l(R_1) + l(R_2)} \prod_{i,j=1,2} \prod_{k,l=1}^{\infty} \frac{a_{i,j} + \hbar(\mu_{i,k} - \mu_{j,l} + l - k)}{a_{i,j} + \hbar(l - k)} \\ &\times \prod_{i=1,2} \prod_{j=1}^{N_f} \prod_{k=1}^{\infty} \frac{\Gamma(\frac{a_i + m_j}{\hbar} + \hbar(1 + \mu_{i,k} - k))}{\Gamma(\frac{a_i + m_j}{\hbar} + \hbar(1 - k))}. \end{aligned} \quad (31)$$

Here the meaning of \tilde{q} is that $\tilde{q} = \Lambda^{4-N_f}$ for $N_f = 1, 2, 3$ and $\tilde{q} = e^{\sqrt{-1}\pi\tau}$ for $N_f = 4$ where τ is the value of the moduli of the Seiberg–Witten curve when $m_1 = \dots = m_4 = 0$.

4 Asymptotic Form of Gopakumar–Vafa Invariants

In this section we derive the asymptotic forms of the Gopakumar–Vafa invariants of the Calabi–Yau toric threefolds that correspond to the $SU(2)$ gauge theory with $N_f = 0, 1, 2, 3, 4$ hypermultiplets. We derive first the asymptotic forms of the Gromov–Witten invariants and then those of the Gopakumar–Vafa invariants.

Let us state the result: for a second homology class $\alpha = d_B[C_B] + d_F[C_F] + d_1[C_{E_1}] + \dots + d_{N_f}[C_{E_{N_f}}]$ with $d_B \geq 1$ and $0 \leq -d_i \leq d_B (1 \leq i \leq N_f)$ ³,

$$n_{\alpha}^g \sim \frac{2^{4d_B+2g-2} \mathcal{F}_{g,d_B}}{(4d_B+2g-3)!} d_F^{4d_B+2g-3} \prod_{i=1}^{N_f} d_B C_{|d_i|} (-1)^{|d_i|}, \quad (32)$$

where ${}_n C_k = \frac{n!}{k!(n-k)!}$ is the binomial coefficient. $\mathcal{F}_{g,k}$ is defined by

$$\log Z_{\text{Nekrasov}}^{(0)} = \sum_{k=1}^{\infty} \Lambda^{4k} \sum_{g=0}^{\infty} (\sqrt{-1}\hbar)^{2g-2} \frac{\mathcal{F}_{g,k}}{a^{4k+2g-2}}. \quad (33)$$

³The reason we deal with the Gopakumar–Vafa invariants at this range is that those for $d_B = 0$ are exactly determined by Z_0 (table 1) and that those at $d_B \geq 1$ are zero unless $0 \leq -d_i \leq d_B (1 \leq i \leq N_f)$ (see previous section).

g	0	1	2	3	4	5	6
k							
1	-2	0	0	0	0	0	0
2	-5	8	-11	14	-17	20	-23
3	-48	$\frac{1024}{3}$	-1872	9376	$-\frac{134608}{3}$	208704	-951440
4	$-\frac{1469}{2}$	13176	-171201	1971646	$-\frac{42777927}{2}$	224106774	-2295588586

Table 2: $2^{4k+2g-2}\mathcal{F}_{g,k}$ for small g, k .

This asymptotic form is valid in the region

$$d_F \gg d_B(g+1). \quad (34)$$

Note that the asymptotic form of the Gopakumar–Vafa invariants consists of two factors. The one is the asymptotic form of $N_f = 0$ case which is a monomial in d_F with the prefactor given by the instanton amplitude of the gauge theory. This factor is common to all the N_f cases and the genus dependence appears only through this part. The other factor is the binomial part which represents the dependence on d_i 's ($1 \leq i \leq N_f$).

For concreteness, we compute $\log \mathcal{Z}_{\text{Nekrasov}}^{(0)}$ up to $\mathcal{O}(\Lambda^{16})$:

$$\begin{aligned}
\log \mathcal{Z}_{\text{Nekrasov}}^{(0)} &= \frac{2\Lambda^4}{\hbar^2 a_{12}^2} \\
&- \frac{\Lambda^8(2\hbar^2 - 5a_{12}^2)}{\hbar^2(\hbar - a_{12})^2 a_{12}^4 (\hbar + a_{12})^2} \\
&+ \frac{16\Lambda^{12}(8\hbar^4 - 26\hbar^2 a_{12}^2 + 9a_{12}^4)}{3\hbar^2(\hbar - a_{12})^2(2\hbar - a_{12})^2 a_{12}^6 (\hbar + a_{12})^2 (2\hbar + a_{12})^2} \\
&- \frac{\Lambda^4(10368\hbar^{10} - 59328\hbar^8 a_{12}^2 + 105356\hbar^6 a_{12}^4 - 67461\hbar^4 a_{12}^6 + 17718\hbar^2 a_{12}^8 - 1469a_{12}^{10})}{2\hbar^2(\hbar - a_{12})^4(2\hbar - a_{12})^2(3\hbar - a_{12})^2 a_{12}^8 (\hbar + a_{12})^4(2\hbar + a_{12})^2(3\hbar + a_{12})^2} \\
&+ \mathcal{O}(\Lambda^{20}).
\end{aligned} \quad (35)$$

If one expand this further by \hbar , one could obtain the coefficients $2^{4k+2g-2}\mathcal{F}_{g,k}$ in table 2.

In the rest of this section we explain the derivation of the asymptotic form. In the previous section, we showed that the topological string amplitudes reproduce Nekrasov's partition functions at the field theory limit (30). By taking the logarithm of the equation

(31), the left-hand side is written as

$$\sum_{d_B=1}^{\infty} q_B^{d_B} \sum_{g=0}^{\infty} g_s^{2g-2} \sum_{d_F, d_1, \dots, d_{N_f}} N_{g, d_B, d_F, d_1, \dots, d_{N_f}} q_F^{d_F} q_1^{d_1} \dots q_{N_f}^{d_{N_f}} \quad (36)$$

On the other hand, the logarithm of the right-hand side takes the following form:

$$\sum_{k=1}^{\infty} \tilde{q}^k \sum_{g=0}^{\infty} (\sqrt{-1}\hbar)^{2g-2} \frac{B_{g,k}^{(N_f)}\left(\frac{m_1}{a}, \dots, \frac{m_4}{a}\right)}{a^{(4-N_f)k+2g-2}}. \quad (37)$$

Here $B_{g,k}^{(N_f)}$ is a polynomial in N_f variables and it has the degree k as a polynomial in each variable. Moreover, the highest degree term is

$$\mathcal{F}_{g,k} \left(\frac{m_1}{a} \dots \frac{m_{N_f}}{a} \right)^k \quad (38)$$

where $\mathcal{F}_{g,k}$ is the same for all $N_f = 0, 1, 2, 3, 4$. This is clear from the expression of Nekrasov's partition function in eq. (31). Therefore it is given by (33). We can also understand this from the well-known fact that a gauge theory with fundamental hypermultiplets reproduces the pure Yang-Mills theory by decoupling hypermultiplets.

Let us adopt the following ansatz:

$$N_{g, d_B, d_F, d_1, \dots, d_{N_f}} \sim r_{d_B}^{(g)}(d_F) \prod_{i=1}^{N_f} C_{|d_i|} (-1)^{|d_i|} \quad (0 \leq -d_i \leq d_B). \quad (39)$$

By substituting the ansatz into (36) and identifying the parameters as (30), we obtain

$$\sum_{d_B=1}^{\infty} \sum_{g=0}^{\infty} \tilde{q}^{d_B} (\sqrt{-1}\hbar)^{2g-2} \beta^{4d_B+2g-2} \sum_{d_F} r_{d_B}^{(g)}(d_F) e^{-2\beta a d_F} \prod_{i=1}^{N_f} (a + m_i)^{d_B}. \quad (40)$$

The last factors have appeared from

$$\prod_{i=1}^{N_f} \sum_{k_i=1}^{d_B} C_{k_i} (-1)^{k_i} q_i^{-k_i} = \prod_{i=1}^{N_f} (1 - q_i^{-1})^{d_B} \sim \prod_{i=1}^{N_f} \beta^{d_B} (a + m_i)^{d_B}.$$

Then comparing (37)(38) and (40) as a series in \hbar and Λ , we obtain the condition which $r_{d_B}^{(g)}(d_F)$ must satisfy:

$$\begin{aligned} \beta^{4k+2g-2} \sum_{d_F} r_{d_B}^{(g)}(d_F) e^{-t_F d_F} &= \frac{\mathcal{F}_{g,k}}{a^{4k+2g-2}} \\ &= (2\beta)^{4k+2g-2} \frac{\mathcal{F}_{g,k}}{t_F^{4k+2g-2}}. \end{aligned} \quad (41)$$

We have actually used only (38) in the comparison and we will discuss this point shortly. In passing to the second line, we have identified a with $t_F/2\beta$. It is clear that the powers of β cancel out and this relation is independent of the limit $\beta \rightarrow 0$. Then we replace the summation over d_F with the integration and regard the left-hand side of (41) as the Laplace transform of $r_{d_B}^{(g)}(d_F)$ as a function in d_F to a function in t_F . Therefore by performing the inverse Laplace transform on the right-hand side, we obtain the following:

$$r_{d_B}^{(g)}(d_F) \sim \frac{2^{4d_B+2g-2}}{(4d_B+2g-3)!} \mathcal{F}_{g,d_B} d_F^{4d_B+2g-3}. \quad (42)$$

This asymptotic form is valid only in the region

$$d_F \gg d_B. \quad (43)$$

In the derivation, we have used only the term with the highest degree in m_1, \dots, m_{N_f} in (37). As it turned out, this is suffice, because the contributions from the terms with lower degrees (in m_i) are smaller: it would be monomials in d_F with degree smaller than $4d_B + 2g - 3$. Thus such terms can be neglected since we consider the region where d_F is large.

Finally let us consider the asymptotic form of the Gopakumar–Vafa invariants. It is the same as the Gromov–Witten invariants, because the contribution from the lower degree and lower genus Gromov–Witten invariants can be neglected. However, the region where the asymptotic form is valid becomes more restricted. It is

$$d_F \gg d_B(g+1). \quad (44)$$

Here we have added the factor $(g+1)$ in the right-hand side of (43) because the number of lower degree/genus terms is $d_B(g+1)$ and this number must be small enough compared to d_F .

5 Example

In this section we explicitly compute the Gromov–Witten invariants and compare them with the asymptotic forms derived in the previous section. We take as an example the case (7)(9) which correspond to the $SU(2)$ gauge theory with $N_f = 2$ fundamental hypermultiplets. The ratios between the Gopakumar–Vafa invariants and the asymptotic forms are shown in figure 4 for $d_B = 1, 2, g = 0, 1, 2$.

We remark that the difference in the topological amplitudes of the two cases appears only in Z_0 and that $Z_{\geq 1}$ are the same. Therefore the distribution of the Gopakumar–Vafa invariants n_{d_B, d_F, d_1, d_2}^g for a homology class $\alpha = d_B[C_B] + d_F[C_F] + d_1[C_{E_1}] + d_2[C_{E_2}]$ are the same when $d_B \geq 1$.

Now we compute the Gopakumar–Vafa invariants of (7)(9) for $d_B = 1, 2$, $g = 0, 1, 2$. Let $G_{d_B}^{(g)}(d_F, d_1, d_2)$ denotes the generating function of the Gopakumar–Vafa invariants for given d_B and g :

$$G_{d_B}^{(g)}(d_1, d_2) := \sum_{d_F, d_1, d_2} n_{d_B, d_F, d_1, d_2}^g q_F^{d_F} q_1^{d_1} q_2^{d_2}. \quad (45)$$

Here the summation over d_F is from zero to infinity and the summations over d_1, d_2 are from $-d_B$ to zero (see section 2). Then $G_{d_B}^{(g)}$'s calculated from the topological string amplitude (15) are as follows.

$$G_1^{(0)} = \frac{q_1 q_2 + q_F - 2q_1 q_F - 2q_2 q_F + q_1 q_2 q_F + q_F^2}{q_1 q_2 (-1 + q_F)^2}, \quad G_1^{(1)} = G_1^{(2)} = 0,$$

$$G_2^{(0)} = \frac{\sum_{i,j=0}^2 f_{ij}(q_F) q_1^i q_2^j}{q_1^2 q_2^2 (-1 + q_F)^6 (1 + q_F)^2},$$

$$f_{00} = -6q_F^4 - 8q_F^5 - 6q_F^6, \quad f_{01} = 5q_F^3 + 15q_F^4 + 15q_F^5 + 5q_F^6,$$

$$f_{02} = -6q_F^3 - 8q_F^4 - 6q_F^5, \quad f_{10} = 5q_F^3 + 15q_F^4 + 15q_F^5 + 5q_F^6,$$

$$f_{11} = q_F - 5q_F^2 - 23q_F^3 - 29q_F^4 - 17q_F^5 - 7q_F^6 - q_F^7 + q_F^8,$$

$$f_{12} = -2q_F + 9q_F^2 + 17q_F^3 + 7q_F^4 + 7q_F^5 + 4q_F^6 - 2q_F^7,$$

$$f_{20} = -6q_F^3 - 8q_F^4 - 6q_F^5,$$

$$f_{21} = -2q_F + 9q_F^2 + 17q_F^3 + 7q_F^4 + 7q_F^5 + 4q_F^6 - 2q_F^7,$$

$$f_{22} = 1 - q_F - 9q_F^2 - 5q_F^3 - 3q_F^4 - 3q_F^5 - q_F^6 + q_F^7$$

$$G_2^{(1)} = \frac{\sum_{i,j=0}^2 f_{ij}(q_F) q_1^i q_2^j}{q_1^2 q_2^2 (-1 + q_F)^8 (1 + q_F)^2},$$

$$f_{00} = 9q_F^5 + 14q_F^6 + 9q_F^7, \quad f_{01} = -8q_F^4 - 24q_F^5 - 24q_F^6 - 8q_F^7,$$

$$f_{02} = 9q_F^4 + 14q_F^5 + 9q_F^6, \quad f_{10} = -8q_F^4 - 24q_F^5 - 24q_F^6 - 8q_F^7,$$

$$f_{11} = 7q_F^3 + 32q_F^4 + 50q_F^5 + 32q_F^6 + 7q_F^7, \quad f_{12} = -8q_F^3 - 24q_F^4 - 24q_F^5 - 8q_F^6,$$

$$f_{20} = 9q_F^4 + 14q_F^5 + 9q_F^6, \quad f_{21} = -8q_F^3 - 24q_F^4 - 24q_F^5 - 8q_F^6,$$

$$f_{22} = 1 - 6q_F + 13q_F^2 + q_F^3 + 37q_F^5 - 14q_F^6 - 8q_F^7 + 13q_F^8 - 6q_F^9 + q_F^{10}$$

$$\begin{aligned}
G_2^{(2)} &= -\frac{q_F^4 \sum_{i,j=0}^2 f_{ij}(q_F) q_1^i q_2^j}{q_1^2 q_2^2 (-1 + q_F)^{10} (1 + q_F)^2}, \\
f_{00} &= 12q_F^2 + 20q_F^3 + 12q_F^4, & f_{01} &= -11q_F - 33q_F^2 - 33q_F^3 - 11q_F^4, \\
f_{02} &= 12q_F + 20q_F^2 + 12q_F^3, & f_{10} &= -11q_F - 33q_F^2 - 33q_F^3 - 11q_F^4, \\
f_{11} &= 10 + 44q_F + 68q_F^2 + 44q_F^3 + 10q_F^4, & f_{12} &= -11 - 33q_F - 33q_F^2 - 11q_F^3, \\
f_{20} &= 12q_F + 20q_F^2 + 12q_F^3, & f_{21} &= -11 - 33q_F - 33q_F^2 - 11q_F^3, \\
f_{22} &= 12 + 20q_F + 12q_F^2.
\end{aligned}$$

By expanding these as series in q_F, q_1^{-1}, q_2^{-1} we obtain the Gopakumar–Vafa invariants listed in table 3.

6 Conclusion

In this article, we have computed the topological string amplitudes of the canonical bundles of toric surfaces which are the Hirzebruch surfaces blown up at $N_f = 0, 1, 2, 3, 4$ points and showed that in a certain limit, they reproduce the Nekrasov’s formulas for $4D$ $\mathcal{N} = 2$ $SU(2)$ gauge theories with N_f fundamental hypermultiplets.

We have also derived the asymptotic form of the Gopakumar–Vafa invariants at all genera from the instanton amplitudes of the gauge theory. From the result in [15] and ours, we expect that the asymptotic form of the Gopakumar–Vafa invariants in the $SU(n + 1)$ cases are given as the product of the two factors: the one is the asymptotic form of $N_f = 0$ case derived in [15] and the other is just the same binomial part as the $SU(2)$ case.

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A Formulas

We list some formulas in this section.

We use letters R, R_i, R', Q etc. for partitions. As mentioned before, $l(R) := \sum_i \mu_i$ for a partition $R = (\mu_1, \mu_2, \dots)$, $\kappa(R) := \sum_i \mu_i(\mu_i - 2i + 1)$ and $d(R)$ is the length of R . $(\mu_{1,i}^\vee)_{i \geq 1}$ (resp. $(\mu_{2,i}^\vee)_{i \geq 1}$) is the conjugate partition of R_1 (resp. R_2) and $(i, j) \in R$ means that there is a box in the place of i -th row and j -th column in the partition R regarded as a Young diagram.

The three point amplitude is $[2, 4, 5, 6]$

$$\begin{aligned} C_{R_1, R_2, R_3} &:= \sum_{Q_1, Q_3} N_{Q_1, Q_3}^{R_1, R_3} q^{\kappa(R_2)/2 + \kappa(R_3)/2} \frac{W_{R_2^t, Q_1} W_{R_2, Q_3^t}}{W_{R_2, \emptyset}} \\ &= (-1)^{l(R_2)} q^{\frac{\kappa R_3}{2}} s_{R_2^t} \sum_Q s_{R_1/Q} (q^{R_2^t + \rho}) s_{R_3^t/Q} (q^{R_2^t + \rho}). \end{aligned} \quad (46)$$

In the first line the summation is over pairs of partitions Q_1, Q_3 . The coefficient $N_{Q_1, Q_2}^{R_1, R_3}$ is defined as follows:

$$N_{Q_1, Q_2}^{R_1, R_2} := \sum_R N_{R, Q_1}^{R_1} N_{R, Q_2}^{R_2}. \quad (47)$$

$N_{R, R''}^{R'''}$ is the tensor product coefficient. In the second line, $s_R, s_{R/Q}$ are the Schur function and the skew Schur function [16]:

$$s_R(x) := \frac{\det(x_i^{\mu_j + d(R) - j})_{1 \leq i, j \leq d(R)}}{\det(x_i^{d(R) - j})_{1 \leq i, j \leq d(R)}}, \quad (48)$$

$$s_{R/Q} := \sum_{R_1} N_{Q, R_1}^R s_{R_1}, \quad (49)$$

where $x = (x_1, \dots, x_{l(R)})$. The tensor product coefficient $N_{R, R''}^{R'''}$ can be computed from the formula

$$s_{R_1} s_{R_2} = \sum_Q N_{R_1, R_2}^Q s_Q \quad (50)$$

or by using the Littlewood-Richardson rule.

The formulas essential to the calculation of $H_{R_1, R_2}^{(k)}$ in section 2 are [16]

$$\sum_Q s_{Q/R_1}(x) s_{Q^t/R_2}(y) = \prod_{i,j=1}^{\infty} (1 + x_i y_j) \sum_Q s_{R_2^t/Q}(x) s_{R_1^t/Q^t}, \quad (51)$$

$$\sum_Q s_{Q/R_1}(x) s_{Q/R_2}(y) = \prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1} \sum_Q s_{R_2/Q}(x) s_{R_1/Q}. \quad (52)$$

Note that the summations in the right-hand sides are infinite but those in the left-hand sides are finite.

Let $f(x)$ be a function in one variable. For a partition $R = (\mu_i)_{i \geq 1}$ ($R^t = (\mu_i^\vee)_{i \geq 1}$), the following identities hold:

$$\prod_{1 \leq i < j \leq \infty} \frac{f(\mu_i - \mu_j + j - i)}{f(j - i)} = \prod_{1 \leq i < j \leq d(R)} \frac{f(\mu_i - \mu_j + j - i)}{f(j - i)} \prod_{i=1}^{d(R)} \prod_{v=1}^{\mu_i} \frac{1}{f(v - i + d(R))} \quad (53)$$

$$= \prod_{(k,l) \in R} \frac{1}{f(\mu_k + \mu_l^\vee - k - l + 1)}. \quad (54)$$

For two partitions $R_1 = (\mu_{1,i})_{i \geq 1}$ and $R_2 = (\mu_{2,i})_{i \geq 1}$, the following identities hold:

$$\begin{aligned} & \prod_{i,j \geq 1} \frac{f(\mu_{1,i} - \mu_{2,j} + j - i)}{f(j - i)} \\ &= \prod_{i=1}^{d(R_1)} \prod_{j=1}^{d(R_2)} \frac{f(\mu_{1,i} - \mu_{2,j} + j - i)}{f(j - i)} \prod_{i=1}^{d(R_1)} \prod_{v=1}^{\mu_{1,i}} \frac{1}{f(v - i + d(R_1))} \prod_{j=1}^{d(R_2)} \prod_{v=1}^{\mu_{2,j}} \frac{1}{f(-v + j - d(R_2))} \end{aligned} \quad (55)$$

$$= \prod_{i,j \geq 1} \frac{f(\mu_{1,i} + \mu_{2,j}^\vee - i - j + 1)}{f(-i - j + 1)} \quad (56)$$

$$= \prod_{(i,j) \in R_1} \frac{1}{f(\mu_{1,i} - j + \mu_{2,j}^\vee - i + 1)} \prod_{(i,j) \in R_2} \frac{1}{f(-\mu_{1,j}^\vee + i - \mu_{2,i} + j - 1)}. \quad (57)$$

The proof of the last expression (57) is the same as the proof of theorem 1.11 in [17]. The proofs of other formulas (53)-(56) can be found in [6].

B Calculation of the integer m_i

In this section, we present the calculation of (1).

By definition, $m_i = \det(v_{\text{in}}, v_{\text{out}})$ where v_{in} and v_{out} are two-component vectors described in figure 5. Note that $v_{\text{out}} \perp v_{i-1}$, $v_{\text{in}} \perp (v_{i+1} - v_i)$ and $\det(v_{i-1}, v_{\text{out}}) = \det(v_{i+1} - v_i, v_{\text{in}}) = 1$. Therefore

$$\det(v_{\text{in}}, v_{\text{out}}) = \det(v_{i+1} - v_i, v_{i-1}) = \det(v_{i+1}, v_{i-1}) - \det(v_i, v_{i-1}). \quad (58)$$

The second term is minus the volume of the triangle spanned by v_i, v_{i-1} , which is -1 because the surface is smooth. To compute the first term, note that the following equation holds by (2):

$$-\gamma_i \det(v_{i+1}, v_i) = \det(v_{i+1}, v_{i-1}).$$

Since $\det(v_{i+1}, v_i) = 1$, the right-hand side is equal to $-\gamma_i$. Therefore

$$m_i = \det(v_{\text{in}}, v_{\text{out}}) = -\gamma_i - 1.$$

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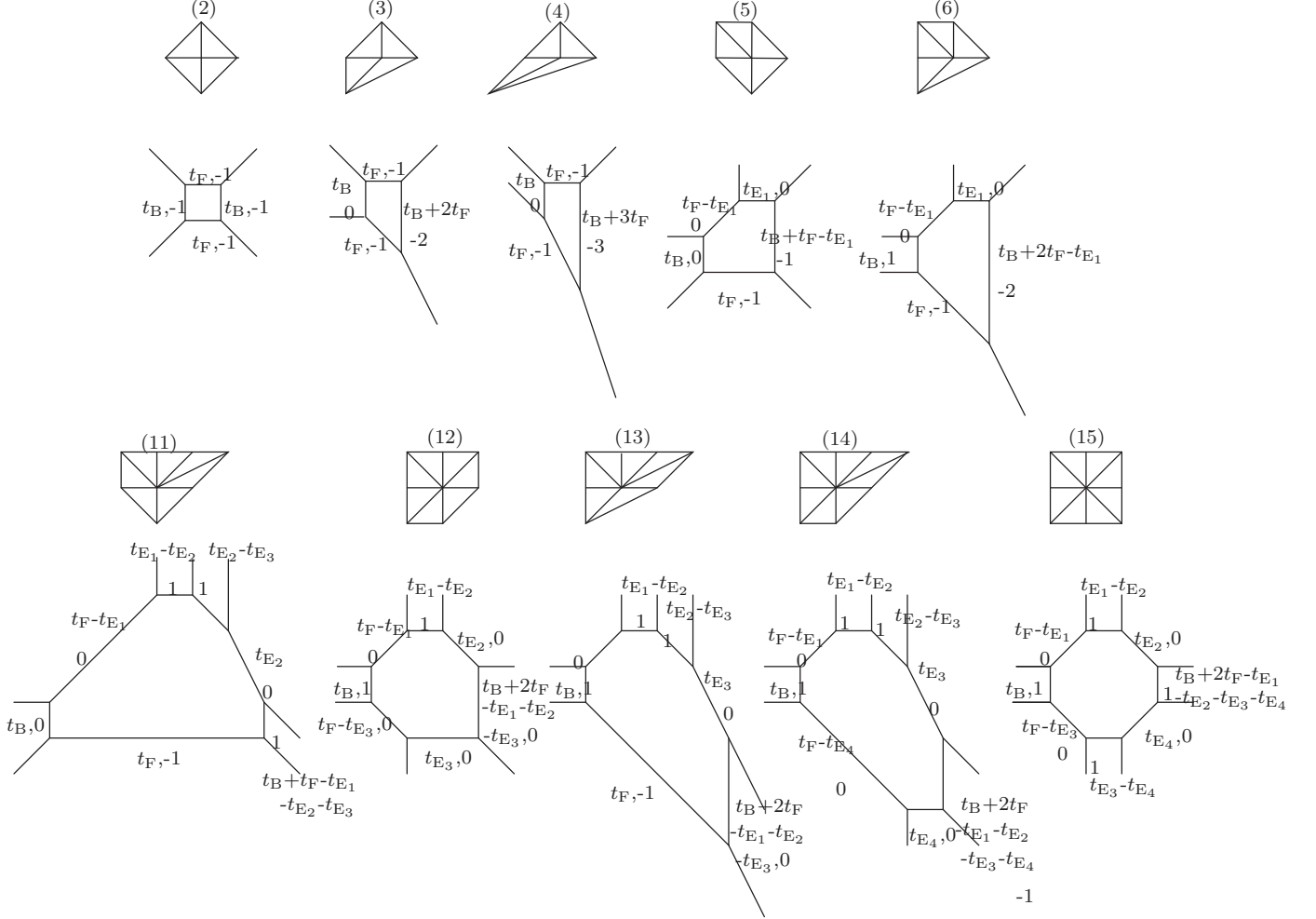


Figure 3: The fans and the web diagrams for the Calabi–Yau toric 3-folds that correspond to four-dimensional $\mathcal{N} = 2$ $SU(2)$ gauge theory with N_f fundamental hypermultiplets. (2)(3)(4) are Hirzebruch surfaces and correspond to $N_f = 0$, (5)(6) to $N_f = 1$, (7)(8)(9) to $N_f = 2$ and (14)(15) to $N_f = 4$. The orientation of the interior edges are taken in the clockwise direction. $N_f = 2$ cases are separately shown in figure 1.

	d_F	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$n_{1,d_F,0,0}^0$		1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41
$n_{1,d_F,0,-1}^0$		0	-2	-4	-6	-8	-10	-12	-14	-16	-18	-20	-22	-24	-26	-28	-30	-32	-34	-36	-38	-40
$n_{1,d_F,-1,0}^0$		0	-2	-4	-6	-8	-10	-12	-14	-16	-18	-20	-22	-24	-26	-28	-30	-32	-34	-36	-38	-40
$n_{1,d_F,-1,-1}^0$		0	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39

	d_F	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n_{2,d_F,0,0}^0$	1	3	-1	-25	-101	-277	-631	-1265	-2323	-3981	-6469	-10057	-15081	-21925	-31051	-42977	
$n_{2,d_F,0,-1}^0$	0	-2	1	29	127	375	898	1876	3554	6252	10375	16423	25001	36829	52752	73750	
$n_{2,d_F,0,-2}^0$	0	0	0	-6	-32	-110	-288	-644	-1280	-2340	-4000	-6490	-10080	-15106	-21952	-31080	
$n_{2,d_F,-1,0}^0$	0	-2	1	29	127	375	898	1876	3554	6252	10375	16423	25001	36829	52752	73750	
$n_{2,d_F,-1,-1}^0$	0	1	-1	-31	-153	-491	-1249	-2731	-5361	-9703	-16481	-26599	-41161	-61491	-89153	-125971	
$n_{2,d_F,-1,-2}^0$	0	0	0	5	35	135	385	910	1890	3570	6270	10395	16445	25025	36855	52780	
$n_{2,d_F,-2,0}^0$	0	0	0	-6	-32	-110	-288	-644	-1280	-2340	-4000	-6490	-10080	-15106	-21952	-31080	
$n_{2,d_F,-2,-1}^0$	0	0	0	5	35	135	385	910	1890	3570	6270	10395	16445	25025	36855	52780	
$n_{2,d_F,-2,-2}^0$	0	0	0	0	-6	-32	-110	-288	-644	-1280	-2340	-4000	-6490	-10080	-15106	-21952	

	d_F	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n_{2,d_F,0,0}^1$	1	0	0	9	68	300	988	2698	6444	13916	27764	51963	92248	156648	256104	405204	
$n_{2,d_F,0,-1}^1$	0	0	0	-8	-72	-352	-1248	-3600	-8976	-20064	-41184	-78936	-143000	-247104	-410176	-657696	
$n_{2,d_F,0,-2}^1$	0	0	0	0	9	68	300	988	2698	6444	13916	27764	51963	92248	156648	256104	
$n_{2,d_F,-1,0}^1$	0	0	0	-8	-72	-352	-1248	-3600	-8976	-20064	-41184	-78936	-143000	-247104	-410176	-657696	
$n_{2,d_F,-1,-1}^1$	0	0	0	7	74	403	1544	4722	12324	28578	60456	118833	219934	387101	652912	1061684	
$n_{2,d_F,-1,-2}^1$	0	0	0	0	-8	-72	-352	-1248	-3600	-8976	-20064	-41184	-78936	-143000	-247104	-410176	
$n_{2,d_F,-2,0}^1$	0	0	0	0	9	68	300	988	2698	6444	13916	27764	51963	92248	156648	256104	
$n_{2,d_F,-2,-1}^1$	0	0	0	0	-8	-72	-352	-1248	-3600	-8976	-20064	-41184	-78936	-143000	-247104	-410176	
$n_{2,d_F,-2,-2}^1$	0	0	0	0	0	9	68	300	988	2698	6444	13916	27764	51963	92248	156648	

	d_F	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n_{2,d_F,0,0}^2$	0	0	0	0	-12	-116	-628	-2488	-8036	-22404	-55836	-127328	-270088	-539416	-1023736	-1859632	
$n_{2,d_F,0,-1}^2$	0	0	0	0	11	121	715	3025	10285	29887	77077	180895	393250	802230	1550978	2863718	
$n_{2,d_F,0,-2}^2$	0	0	0	0	0	-12	-116	-628	-2488	-8036	-22404	-55836	-127328	-270088	-539416	-1023736	
$n_{2,d_F,-1,0}^2$	0	0	0	0	11	121	715	3025	10285	29887	77077	180895	393250	802230	1550978	2863718	
$n_{2,d_F,-1,-1}^2$	0	0	0	0	-10	-124	-800	-3620	-12980	-39380	-105248	-254540	-567710	-1184040	-2333760	-4382872	
$n_{2,d_F,-1,-2}^2$	0	0	0	0	0	11	121	715	3025	10285	29887	77077	180895	393250	802230	1550978	
$n_{2,d_F,-2,0}^2$	0	0	0	0	0	-12	-116	-628	-2488	-8036	-22404	-55836	-127328	-270088	-539416	-1023736	
$n_{2,d_F,-2,-1}^2$	0	0	0	0	0	11	121	715	3025	10285	29887	77077	180895	393250	802230	1550978	
$n_{2,d_F,-2,-2}^2$	0	0	0	0	0	0	-12	-116	-628	-2488	-8036	-22404	-55836	-127328	-270088	-539416	

Table 3: The Gopakumar–Vafa invariants of (7)(9) for $(d_B, g) = (1, 0), (2, 0), (2, 1), (2, 2)$. Those with $(d_B, g) = (1, 1), (1, 2)$ are omitted because they are zero. Note that the Gopakumar–Vafa invariants are symmetric with respect to d_1, d_2 .

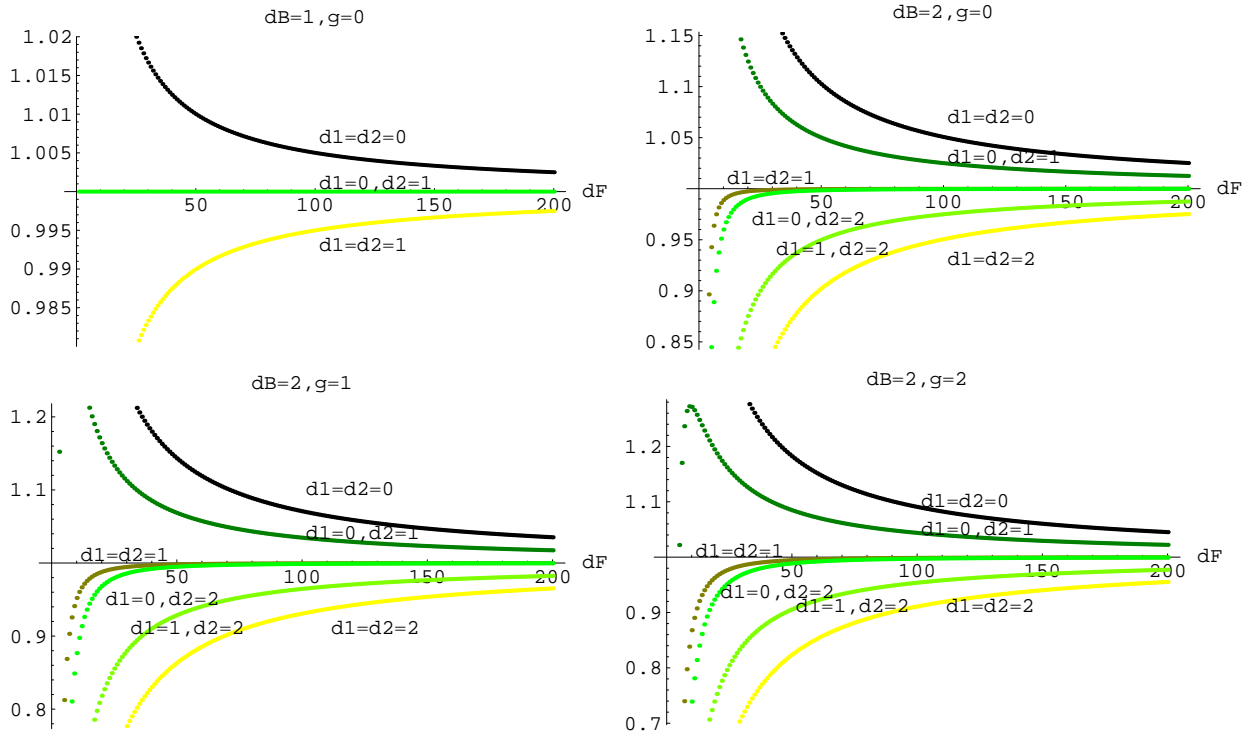


Figure 4: The ratio between the Gopakumar–Vafa invariants n_{d_B, d_F, d_1, d_2}^g and the asymptotic form in the case of (7)(9) for $(d_B, g) = (1, 0), (2, 0), (2, 1), (2, 2)$.

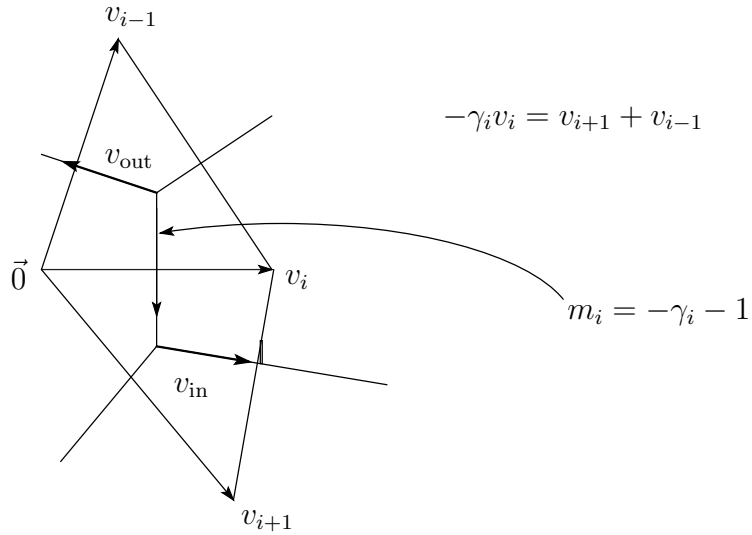


Figure 5: $m_i = \det(v_{\text{in}}, v_{\text{out}}) = -\gamma_i - 1$.